Robust Finite-Element Solution of Lossy and Unbounded Electromagnetic Eigenvalue Problems

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***Abstract*—An efficient algorithm is presented for the finite-element solution of electromagnetic eigenvalue problems associated with lossy and unbounded structures. The algorithm is based on the $\vec{E}-\vec{B}$ formulation of the finite-element approximation of the electromagnetic equations. The special relationship between the vector bases used for the expansion of the electric field vector $\vec{E}$ and the magnetic flux density vector $\vec{B}$ is used to reduce the computational complexity of the proposed formulation. The occurrence of spurious dc modes is avoided through the careful selection of divergence-free initial vectors in the Lanczos–Arnoldi-based iterative schemes used by the proposed algorithm. The resultant algorithm is only marginally more expensive than standard finite-element-based algorithms used for electromagnetic eigenvalue problems involving lossless structures. Numerical experiments from the application of the proposed algorithm to the eigenvalue analysis of both lossless and lossy cavities are used to demonstrate its accuracy, computational efficiency, and robustness.**

***Index Terms*—Arnoldi algorithm, eigenvalue problems, finite-element method, Lanczos algorithm, lossy media, unbounded media.**

I. Introduction

DUE TO ITS superior geometry and material modeling versatility, the finite-element method (FEM) has become one of the most dominant numerical methods for the eigenanalysis of three-dimensional (3-D) lossless inhomogeneous cavities [1]–[3]. However, its application to lossy electromagnetic eigenvalue problems, where the loss may be either due to the finite conductivity of the material or the electromagnetic energy leakage in the case of unbounded geometries, has not been as straightforward. The reason for this is that the presence of loss leads to nonlinear generalized eigenvalue problems. To address this complication, methodologies based on the subspace iteration method [4] and the transformation of the nonlinear problems to the linear ones [5] have been proposed.

In this paper, the $\vec{E}-\vec{B}$ finite-element formulation of [7] is proposed as yet another means to address efficiently the aforementioned difficulty. This new formulation is based on the finite-element discretization of the coupled system of Maxwell's curl equations. Yet, its computational efficiency is not hindered by the fact that both the electric field $\vec{E}$ and the magnetic flux density $\vec{B}$ are discretized. The reason for this is a special relationship between the tangentially continuous and the normally continuous vector spaces used, respectively, for the discretization of $\vec{E}$ and $\vec{B}$ [6] that is exploited to make the computational complexity of this new formulation equivalent to that of the conventional finite-element formulation of the vector-wave equation.

One of the applications of the $\vec{E}-\vec{B}$ formulation has been the model order reduction of discrete approximations of electromagnetic boundary value problems for the purpose of compact macromodeling of electromagnetic devices [8]. Model order reduction is closely related to the problem of eigendecomposition of the governing system of state equations describing the discrete electromagnetic problem. Therefore, it is only reasonable to consider the application of the $\vec{E}-\vec{B}$ formulation to the FEM eigenanalysis of 3-D electromagnetic problems involving lossy and/or unbounded structures.

The paper is organized as follows. Sections II–V are devoted to the mathematical development of the proposed methodology and the construction of the associated algorithms. Emphasis is placed on the development of effective means for the removal of the spurious dc modes, which are known to occur and hinder numerical convergence and algorithm robustness. Following the mathematical development, numerical experiments are presented in Section VI to demonstrate the computational efficiency of the proposed algorithm and its effectiveness in eliminating spurious dc modes and, thus, extracting accurately eigenfrequencies irrespective of their proximity to the troublesome $f \sim 0$ regime. The paper concludes with a summary of the key attributes of the proposed eigenvalue solvers.

II. Finite-Element Model

For the purpose of interest, the computational domain occupies a region of space Ω, which is source free and, in the general case, unbounded. Assuming the presence of ohmic loss in the media, the electric field satisfies the vector Helmholtz equation

$$\nabla \times \frac{1}{\mu} \nabla \times \vec{E} + j\omega\sigma\vec{E} - \omega^2\epsilon\vec{E} = 0. \tag{1}$$

The finite-element approximation of (1) is effected through the expansion of $\vec{E}$ in terms of tangentially continuous vector basis functions $\vec{w}_t$ associated with the elements of a finite-element grid used to discretize the region, and the subsequent development of the weak statement of the above equation in the Galerkin form through its testing with each of the expansion functions. The resulting FEM system has the form

$$\left(S + j\omega Z - \omega^2 T\right)x_e = 0 \tag{2}$$

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where x_e is the vector containing the expansion coefficients of $\vec{E}$, and the elements of the matrices S, Z, T are given by

$$S_{i,j} = \int_\Omega \nabla \times \vec{w}_i \cdot \frac{1}{\mu} \nabla \times \vec{w}_j \, dv$$
$$T_{i,j} = \int_\Omega \vec{w}_i \cdot \epsilon \vec{w}_j \, dv$$
$$Z_{i,j} = \oint_{\partial\Omega_1} \hat{n} \times \vec{w}_i \cdot \frac{1}{\eta} \hat{n} \times \vec{w}_j \, ds + \int_\Omega \vec{w}_i \cdot \sigma \vec{w}_j \, dv. \quad (3)$$

In the above, $\partial\Omega_1$ is the unbounded portion of the boundary of Ω. Grid truncation is effected through the enforcement of a first-order radiation boundary condition, which is physically described in terms of the imposition of a surface impedance on $\partial\Omega_1$ of value equal to the intrinsic impedance η of the unbounded portion of the region.

For the lossless case, the Z matrix is zero and (3) reduces to a linear generalized eigenvalue problem that can be solved directly using the Lanzcos or Arnoldi algorithms. In the general case of a lossy and/or unbounded region, (2) is a nonlinear eigenvalue problem.

For the case of the $\vec{E} - \vec{B}$ formulation [7], both Maxwell's curl equations

$$\nabla \times \vec{E} = -j\omega\vec{B}$$
$$\nabla \times \left(\vec{B}\Big/\mu\right) = j\omega\epsilon\vec{E} + \sigma\vec{E} \quad (4)$$

are discretized. Due to the boundary condition properties of $\vec{E}$ and $\vec{B}$, $\vec{E}$ is expanded in terms of tangentially continuous vector basis functions $\vec{w}_t$, while $\vec{B}$ is expanded in terms of normally continuous vector basis functions $\vec{w}_n$ [6]. The weak statement is obtained by testing the two equations using $\vec{w}_n$ and $\vec{w}_t$, respectively. The resulting FEM system assumes the form

$$\begin{bmatrix} 0 & jD \\ -jD^T & jZ \end{bmatrix} \begin{bmatrix} x_b \\ x_e \end{bmatrix} = \omega \begin{bmatrix} P & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} x_b \\ x_e \end{bmatrix}. \quad (5)$$

In the above equations, x_e and x_b are the vectors containing the expansion coefficients for $\vec{E}$ and $\vec{B}$, respectively, Z and T are the matrices appearing in (3), and the elements of the new matrices D and P are given by

$$D_{i,j} = \int_\Omega \vec{w}_{n,i} \cdot \frac{1}{\mu} \nabla \times \vec{w}_{t,j} \, dv$$
$$P_{i,j} = \int_\Omega \vec{w}_{n,i} \cdot \frac{1}{\mu} \vec{w}_{n,j} \, dv. \quad (6)$$

In compact form, (5) is written as follows:

$$(G - \omega C)x = 0 \quad (7)$$

where $x = (x_b^T, x_e^T)^T$. Clearly, the resulting matrix eigenvalue problem is linear; hence, this formulation is most suitable for the development of the eigensolver for the general case of lossy media.

III. Algorithm for Lossless Media

The lossless case is considered first. From (2), the FEM eigenvalue equation assumes the simple form $Sx_e = \omega^2 T x_e$. Krylov subspace eigenvalue algorithms such as Lanczos or Arnoldi are very successful in capturing the dominant poles with very good accuracy well before the generation of the entire Krylov space. However, the dominant eigenvalues found by these algorithms are usually the ones with the largest magnitude, while the desired ones for electromagnetic cavities are the ones associated with the low-frequency modes. Therefore, the spectral transformation [9]

$$Sx_e = \omega^2 T x_e \;\rightarrow\; Ax_e = \frac{1}{\delta} x_e \quad (8)$$

where $\delta = \omega^2 - \omega_0^2$ and $A = (S - \omega_0^2 T)^{-1} T$ is used to map the eigenvalues of interest (close to the expansion point ω_0^2) to those with largest magnitude, while the corresponding eigenvectors remain unchanged.

The main procedure of a Krylov subspace method is the recursive construction of the orthonormal Krylov subspace

$$Kr_n(v_0, Av_0, \ldots, Av_0^{n-1}) = (v_0, v_1, \ldots, v^{n-1}) = V_n \quad (9)$$

where v_0 is an initial vector. Proper selection of the initial vector is important to ensure convergence and is discussed in Section V. The recursive process used for the construction can be written in matrix form as follows:

$$AV_n = V_n H_n + \tilde{v}_{n+1} e_n^T \quad (10)$$

where $e_n^T = (0, \ldots, 0, 1)_{1\times n}$, and H_n is a $n \times n$ square matrix.

When S and T are symmetric, it is easily seen that A is symmetric with respect to the inner product $\langle x, y\rangle = x^H T y$. Thus, the Lanczos algorithm can be used with the benefit of shorter recurrences. In this case, H_n is an n by n tridiagonal matrix with entries determined by the following recurrence relation:

$$\tilde{v}_{i+1} = Av_i - h_{i-1,i} v_{i-1} - h_{i,i} v_i \quad (11)$$

where

$$h_{i-1,i} = \langle v_{i-1}, Av_i \rangle$$
$$h_{i,i} = \langle v_i, Av_i \rangle$$
$$h_{i+1,i} = \|\tilde{v}_{i+1}\|$$
$$v_{i+1} = \tilde{v}_{i+1}/h_{i+1,i}. \quad (12)$$

Let (s, λ) denote an eigenpair of H_n, i.e., $H_n s = \lambda s$ and $x = V_n s$. It is then

$$Ax - \lambda x = AV_n s - V_n H_n s = \left(e_n^T s\right)\tilde{v}_{n+1}. \quad (13)$$

Upon convergence, the calculated eigenpair is $(V_n s, \lambda)$. Thus, a stopping criterion for the algorithm can be the following:

$$\frac{\|Ax - \lambda x\|}{\|\lambda x\|} = \frac{\|(e_n^T s)\| \|\tilde{v}_{n+1}\|}{\|\lambda V_n s\|} < \rho \quad (14)$$

where ρ defines the desired accuracy.

Although the Lanczos process has shorter recurrence than the Arnoldi algorithm, it suffers from loss of orthogonality. Typically, such loss of orthogonality in the bases of the generated Krylov subspace occurs when the number of iterations is large. As will be shown in the numerical results section, loss of orthogonality leads to erroneous eigenvalues and, thus, needs to be avoided. Toward this objective, several enhancements of the

standard Lanczos algorithm have been proposed over the past few years [11]. These enhancements come at the cost of increased algorithmic complexity.

Loss of orthogonality is not the case for the Arnoldi process. Although its recurrence is computationally more expensive than that of Lanczos, the compensating effect of its fast convergence, combined with its algorithmic simplicity, make it very attractive as an alternative to the Lanczos process. Furthermore, symmetry is not required when the Arnoldi process is used. Thus, the inner product is taken in the usual manner, i.e., $\langle x, y\rangle = x^H y$. In the Arnoldi process, the matrix H_n is an $n \times n$ upper Hessenberg matrix with entries obtained through the following recursive equation:

$$\tilde{v}_{i+1} = Av_i - h_{i,i}v_i - h_{i-1,i}v_{i-1} \cdots - h_{0,i}v_0 \tag{15}$$

where

$$\begin{aligned} h_{j,i} &= \langle v_j, Av_i\rangle \\ j &= i, i-1, \ldots, 0 \\ h_{i+1,i} &= \|\tilde{v}_{i+1}\| \\ v_{i+1} &= \tilde{v}_{i+1}/h_{i+1,i}. \end{aligned} \tag{16}$$

Similar to Lanczos, the eigenpairs of (2) are determined from those of H_n. Equation (14) is still used to monitor convergence.

The extension of the aforementioned algorithms to block algorithms for the calculation of multiple eigenvalues is rather straightforward. Furthermore, the deflation technique [3] can be used. Deflation takes advantage of the orthogonality of two different eigenvectors $x_{e,i}$ and $x_{e,j}$

$$x_{e,i}^H S x_{e,j} = x_{e,i}^H T x_{e,j} = 0, \qquad i \neq j \tag{17}$$

to remove those eigenvectors that have already converged during the application of the Lanczos–Arnoldi process for the calculation of the next eigenvector [3]. This is effected by means of the following operation:

$$\tilde{v}_{i+1} \leftarrow \tilde{v}_{i+1} - \frac{x_e^H T \tilde{v}_{i+1}}{x_e^H T x_e} x_e. \tag{18}$$

Theoretically, this operation needs to be done once for the initial vector. However, because of roundoff error, its application for each and every Lanczos–Arnoldi vector is strongly recommended.

For electromagnetic eigenvalue problems involving lossless media, either the Lanczos or the Arnoldi process can be applied to the matrix statement of the discrete problem obtained through the $\vec{E}$ formulation. For our purposes, and for the reasons mentioned above, the Arnoldi process is used exclusively. The cost of the numerical computation is dominated by the solution of the matrix equation $(S - \omega_0^2 T)$ at each iteration. Depending on the size of the problem, the solution is obtained either through LU factorization or through the application of an iterative solver.

IV. Algorithm for Lossy Media

When loss is present, either in the form of loss in the media or radiation loss, the $\vec{E} - \vec{B}$ formulation (5) is used instead. Once again, a spectral transformation is used to improve convergence. Hence, the matrix statement of the discrete eigenvalue problem becomes

$$(G - \omega C)x = 0 \;\rightarrow\; Ax = \frac{1}{\delta}x \tag{19}$$

where $\delta = \omega - \omega_0$ and

$$A = (G - \omega_0 C)^{-1} C. \tag{20}$$

Compared with the matrix obtained from the $\vec{E}$ formulation, the dimension of the resulting matrix is almost doubled; hence, at first appearance, the computational cost of the eigenanalysis of lossy electromagnetic problems is significantly higher than that for lossless ones. However, an algorithm is presented below that enables the solution of (19) at a cost only marginally higher than that for the lossless case.

Consider the generation of the next Arnoldi vector v_{i+1} as follows:

$$\tilde{v}_{i+1} = Av_i - h_{i,i}v_i - h_{i-1,i}v_{i-1} \cdots - h_{0,i}v_0. \tag{21}$$

The main computation is the matrix–vector product

$$Av_i = (G - \omega_0 C)^{-1} C v_i = p. \tag{22}$$

With the vectors p and v_i split into their electric field and magnetic flux density parts

$$p = \begin{bmatrix} p_b \\ p_e \end{bmatrix} \quad v_i = \begin{bmatrix} v_b \\ v_e \end{bmatrix}. \tag{23}$$

One has

$$Av_i = p \rightarrow \begin{bmatrix} -\omega_0 P & jD \\ -jD^T & jZ - \omega_0 T \end{bmatrix} \begin{bmatrix} p_b \\ p_e \end{bmatrix} = \begin{bmatrix} Pv_{i,b} \\ Tv_{i,e} \end{bmatrix}. \tag{24}$$

The resulting matrix equation can be split into two equations. The first equation concerns the calculation of the expansion coefficients for $\vec{E}$

$$\left(D^T P^{-1} D + j\omega_0 Z - \omega_0^2 T\right) p_e = -jD^T v_{i,b} + \omega_0 T v_{i,e}. \tag{25}$$

Subsequently, the expansion coefficients for $\vec{B}$ are calculated from the second equation

$$p_b = \frac{1}{\omega_0} jP^{-1} D p_e - \frac{1}{\omega_0} v_{i,b}. \tag{26}$$

At this point, it is appropriate to discuss briefly the relationship between tangentially and normally continuous vector spaces. Recall that the curl of the tangentially continuous vector space is a subset of the normally continuous vector space [6], i.e., $\nabla \times W_t \subset W_n$. Hence, a linear matrix relationship exists between the basis functions of W_t and W_n as follows:

$$\left[\nabla \times \vec{w}_{t1}, \ldots, \nabla \times \vec{w}_{tN_t}\right] = \left[\vec{w}_{n1}, \ldots, \vec{w}_{nN_n}\right] Y_{N_n \times N_t}. \tag{27}$$

Due to the bilinear form of the FEM matrices, it is straightforward to show that

$$D^T P^{-1} D = S \quad P^{-1} D = Y. \tag{28}$$

In view of the above results, it is apparent that, in the application of (25) and (26), the matrix $D^T P^{-1} D$ is already available from the FEM approximation of the vector Helmholtz equation for the electric field.

Y, which is called the circulation matrix [12], does not need to be calculated explicitly. To explain, this matrix relates field to flux. The product of a vector with this matrix is effected through a localized operation, namely, by summing up the three edge expansion coefficients associated with a facet for each facet in the computational domain.

Thus, the final issue to be addressed is how to avoid the generation of the matrix D. Toward this objective, the following transformation is introduced for both p and v_i:

$$\begin{bmatrix} p_b \\ p_e \end{bmatrix} = \begin{bmatrix} jP^{-1}D & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p_{bb} \\ p_e \end{bmatrix}$$
$$\begin{bmatrix} v_{i,b} \\ v_{i,e} \end{bmatrix} = \begin{bmatrix} jP^{-1}D & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v_{i,bb} \\ v_{i,e} \end{bmatrix}. \tag{29}$$

In view of the above transformation, the update of the next Arnoldi vector is performed as follows:

$$\begin{aligned} \left(S + j\omega_0 Z - \omega_0^2 T\right) p_e &= S v_{i,bb} + \omega_0 T v_{i,e} \\ p_{bb} &= \frac{1}{\omega_0}\left(p_e - v_{i,bb}\right) \\ p_b &= jY p_{bb}. \end{aligned} \tag{30}$$

To summarize, the application of the Arnoldi process to (19) does not require the construction of any matrices other than the ones associated with the finite-element approximation of the vector Helmholtz equation for the electric field. Furthermore, due to the splitting of the generated Arnoldi vectors into their electric field and magnetic flux density parts and the updating process of (30), the computational complexity of the resulting modified algorithm is only slightly higher than that for the lossless case. From the point-of-view of memory requirements, the auxiliary vectors V_{bb} have to be stored in addition to the Arnoldi vectors (V_b, V_e). However, due to the fast convergence of the Arnoldi algorithm, the associate memory overhead is not significant since only a small number of vectors are generated.

The robustness of the Arnoldi process can be improved further by balancing the magnitudes of the electric field part x_e and the magnetic flux density part x_b of the Arnoldi vectors. The magnetic flux density is multiplied by ω_0 and the generalized eigenvalue problem assumes the form

$$\underbrace{\begin{bmatrix} 0 & jD \\ -\dfrac{jD^T}{\omega_0} & jZ \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} \omega_0 x_b \\ x_e \end{bmatrix}}_{x} = \omega \underbrace{\begin{bmatrix} \dfrac{P}{\omega_0} & 0 \\ 0 & T \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \omega_0 x_b \\ x_e \end{bmatrix}}_{x}. \tag{31}$$

Consequently, the update equations for the generation of the Arnoldi vector become

$$\begin{aligned} \left(S + j\omega_0 Z - \omega_0^2 T\right) p_e &= \frac{1}{\omega_0} S v_{i,bb} + \omega_0 T v_{i,e} \\ p_{bb} &= p_e - \frac{v_{i,bb}}{\omega_0} \\ p_b &= jY p_{bb}. \end{aligned} \tag{32}$$

Finally, it is pointed out that, in the presence of loss, the lack of orthogonality between different eigenmodes necessitates the use of a block algorithm for the simultaneous calculation of several eigenmodes.

A pseudocode description of the resulting algorithm is as follows.

```
1. Select initial vector v_{0,e}; set v_{0,b} ← 0,
   v_{0,bb} ← 0;
2. i ← 0;
3. p ← Av_i;      // calculate p_e, p_bb, p_b
   using (32)
     3.a. Calculate p_e;
     3.b. Remove dc modes in p_e;
     3.c. Calculate p_bb and p_b.
4. Update H and calculate v_{i+1} using (15)
   and (16);
5. Calculate all the eigenpairs (s, λ) of
   H;
6. For each eigenpair (s, λ), estimate
   residual using (14);
7. If the residual of an eigenpair (s, λ)
   is smaller than ρ, output (V_e s, λ) and
   stop; else i ← i + 1 and go to 3.
```

The difference between the algorithms for lossy and lossless media lies in the way the Arnoldi vector is updated in step 3. As already mentioned, the selection of the initial vector $v_{0,e}$ requires care. Also, spurious dc modes need to be removed during the iteration process (step 3.b). These two important issues are considered below.

V. Elimination of DC Modes

Even though the use of a tangentially continuous vector space instead of a scalar space for the expansion of $\vec{E}$ can eliminate spurious ac modes, it introduces spurious dc modes [3]. This is due to the fact that the tangentially continuous vector space W_t contains the gradient of the scalar space Φ [6]. Thus, there exists a matrix G_r such that

$$\begin{aligned} \nabla\Phi = \vec{W}_t G_r &\rightarrow \left[\nabla\phi_1, \ldots, \nabla\phi_{N_s}\right] \\ &= \left[w_{t,1}, \ldots, w_{t,N_t}\right] G_{r\,N_t \times N_s} \end{aligned} \tag{33}$$

where G_r is the gradient matrix defined in [10]. From the definition of the matrices S and D and their bilinear forms, it is straightforward to deduce the following results:

$$G_r^T S = 0 \quad S G_r = 0 \quad D G_r = 0. \tag{34}$$

In view of the second of the above equations and the matrix eigenvalue statement (2) for the case of lossless media where $Z = 0$, it is immediately evident that G_r contains all the dc modes. Unless these dc modes are eliminated, the Lanczos–Arnoldi process produces a large amount of eigenvalues with magnitude several orders less than the desired ones. The presence of these dc modes slows down convergence and wastes memory.

Since the dc modes do not satisfy the divergence-free condition at $\omega = 0$, they can be eliminated through the imposition of the divergence-free requirement on every generated Lanczos–Arnoldi vector. In the ideal case of zero roundoff error, the divergence-free condition need only be imposed on the initial vectors. Such a divergence-free initial vector can be chosen to be the electric field induced by a loop current source. However, roundoff error is unavoidable; thus, the selection of divergence-free initial vectors alone is not sufficient to guarantee removal of the spurious dc modes. The error due to the presence of these modes is most severe when the expansion frequency ω_0 assumes low values.

To address this difficulty, the technique suggested in [3] can be used. The basic idea is to add to the electric-field vector p_e, a vector $q_e = G_r\Psi$, where Ψ is a vector to be determined such that the spurious dc modes are canceled. The new vector

$$p_e \leftarrow p_e + G_r\Psi \tag{35}$$

then satisfies the divergence-free condition at dc ($\nabla \cdot \vec{D} = 0$). In matrix form, this becomes

$$M\Psi = -G_r^T T p_e \tag{36}$$

where M is the finite-element matrix for the corresponding electrostatic problem

$$M_{i,j} = \int_\Omega \nabla\phi_i \cdot \epsilon\nabla\phi_j \, dv \tag{37}$$

with ϕ_i and ϕ_j being scalar basis functions. The aforementioned vector Ψ is the solution to this electrostatic problem. Once found, (35) is used to eliminate the spurious dc modes.

For the case of lossy media, not every column vector in G_r corresponds to a spurious dc mode of the matrix eigenvalue problem. This is evident from the product of $[0, G_r^T]^T$ with the matrix G in (31). Use of the results in (34) yields

$$\begin{bmatrix} 0 & jD \\ -\dfrac{jD^T}{\omega_0} & jZ \end{bmatrix} \begin{bmatrix} 0 \\ G_r \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & jZG_r \end{bmatrix}. \tag{38}$$

Consequently, the dc modes are those column vectors of G_r for which the multiplication with the matrix Z yields the null vector. To elaborate, each column vector in G_r corresponds to a node in the computational domain. Furthermore, the nonzero entries in the column vector correspond to an edge connected to the node. Thus, the spurious dc modes are associated with those nodes that are neither inside the portion of the computational domain that contains the lossy media, nor on the surface of an absorbing boundary. Therefore, the correction through the electrostatic solution of (36) is only performed in the lossless region.

In summary, the spurious dc modes can be eliminated by keeping the generated Lanczos–Arnoldi vectors divergence free. This is achieved by first making sure that the initial vectors are divergence free. For this purpose, the fields generated by closed electric-current loops are used as initial vectors. In addition, during the Lanczos–Arnoldi process, the solution of (36) is utilized through the correction operation of (35) in order to ensure the elimination from the constructed Lanczos–Arnoldi vectors of any spurious dc modes that may be introduced because of roundoff error. For the case of lossy media, G_r and M are only constructed for the nodes in the computational domain exclusive of the lossy media and absorbing boundary.

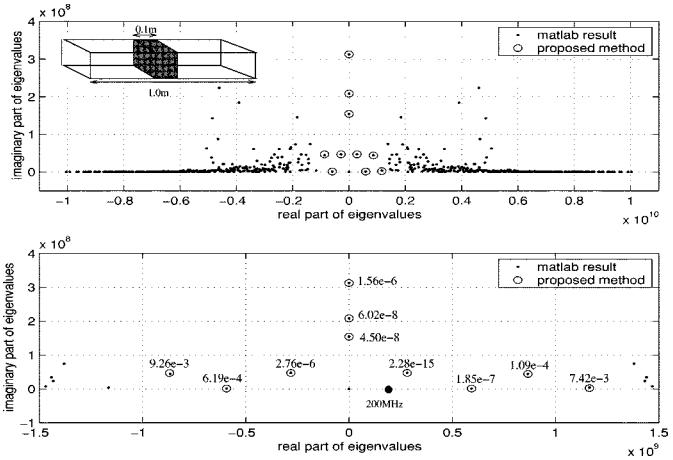

Fig. 1. Eigenfrequencies of a parallel-plate waveguide with a lossy dielectric block in the middle. The top plot indicates the eiegenfrequencies. In the bottom plot, the associated residuals are indicated.

VI. Applications

In the following, the proposed methodologies and algorithms are implemented for the solution of electromagnetic eigenvalue problems associated with both lossless and lossy media. All calculations are performed on a Pentium III 600-MHz PC with double precision arithmetic. The tolerance is set to $\rho = 1.0e^{-6}$.

The first example is the parallel-plate waveguide with a lossy dielectric block in the center, as shown in the insert of Fig. 1. The cross section is square of side 0.1 m. The top and bottom plates are perfectly conducting. The side walls are assumed to be perfect magnetic walls. The two ports are terminated by perfect conductors. The relative dielectric constant of the dielectric block is 2.0 and its conductivity is 0.1 S/m. The $\vec{E} - \vec{B}$ formulation is generated explicitly, and is exported to MATLAB to perform the complete eigenvalue analysis for reference. The block Arnoldi algorithm described in Section IV with spurious dc-mode removal is also used for the calculation of the three dominant modes. The expansion frequency point is taken to be $f_0 = 200$ MHz. After 11 iterations, the calculated eigenfrequencies with residual less than $1.0e^{-2}$ are plotted in the top plot in Fig. 1. The corresponding residuals are shown in the bottom plot. Clearly, the eigenfrequencies close to the expansion point are calculated first. Furthermore, the zero eigenfrequencies are successfully avoided.

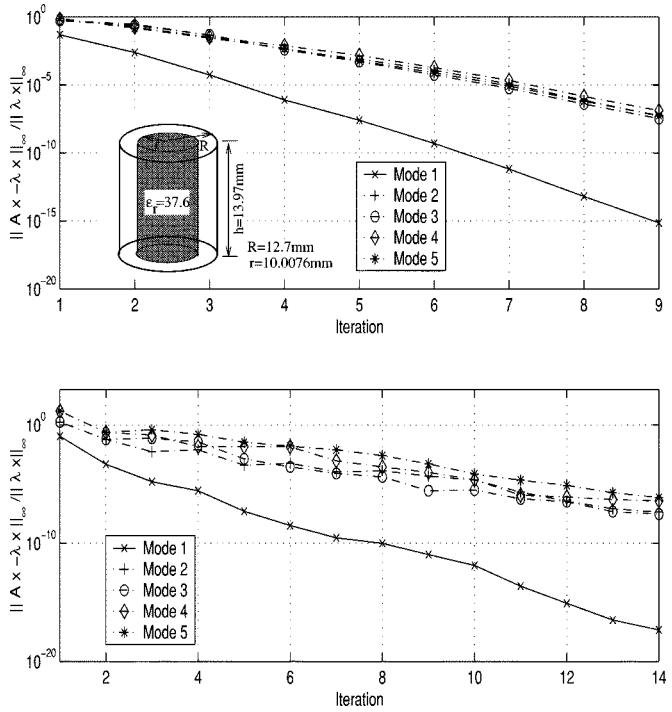

Fig. 2. Convergence of the five dominant modes of a partially filled PEC cylindrical cavity of circular cross section. The top plot is for the case of lossless dielectric filling. The bottom plot is for the case of lossy dielectric filling. In both cases, a block Arnoldi algorithm is used.

TABLE I
LOWEST EIGENFREQUENCIES OF PARTIALLY FILLED LOSSLESS CYLINDRICAL CAVITY OBTAINED USING A BLOCK ARNOLDI PROCESS

Mode	Analyt.	Numerical	Residual	Error
1	1498	1500.0	1.008e-14	0.13%
2	2435	2441.5	4.149e-08	0.27%
3	2435	2442.4	5.704e-08	0.31%
4	2504	2505.1	9.873e-08	0.04%
5	2504	2505.6	2.304e-07	0.06%
6	3029	3045.6	8.523e-05	0.55%
7	3331	3334.4	3.163e-04	0.10%

The second example is a circular cylindrical cavity, partially filled with a dielectric rod of relative permittivity $\epsilon_r = 37.6$. The cavity walls are assumed to be perfect electric conductors (PECs). The geometry is shown in the insert of Fig. 2. The average grid size in the FEM mesh is 2.0 mm, and the number of degrees of freedom is 5748. The objective is to extract the five lowest eigenmodes. Thus, the expansion frequency is taken to be $f_0 = 1.2$ GHz.

First, the dielectric rod is assumed to be lossless. The block Arnoldi algorithm is used to capture the first five dominant modes. The calculated resonant frequencies of the seven lowest eigenmodes are compared to their analytical ones in Table I. The solution required nine iterations for the five modes to converge. The total CPU time was 61.8 s. No spurious dc modes were generated. Furthermore, every calculated mode corresponds to an analytical one, indicating that no spurious ac modes occur either. The top plot in Fig. 2 depicts the convergence behavior in the calculation of the first five eigenfrequencies. The first (lowest) eigenfrequency converges much faster, while the other four exhibit a slightly slower convergence, requiring nine iterations to reach the tolerance threshold.

TABLE II
LOWEST EIGENFREQUENCIES OF PARTIALLY FILLED LOSSLESS CYLINDRICAL CAVITY OBTAINED USING A BLOCK LANCZOS PROCESS

Mode	Analyt.	Numerical	Residual	Spurious
1	1498	1500.0	2.265e-06	
2	1498	1500.0	1.551e-05	Yes
3	2435	2441.5	5.876e-03	
4	2435	2442.4	6.631e-04	
5	2435	2442.2	1.532e-01	Yes
6	2504	2505.1	1.762e-05	
7	2504	2505.6	4.531e-05	

TABLE III
EIGENFREQUENCIES OF THE LOWEST MODES OF A PARTIALLY FILLED LOSSY CYLINDRICAL CAVITY OBTAINED USING A BLOCK ARNOLDI PROCESS

Mode	Calculated	Residual
1	1499.9+j23.86	4.550e-18
2	2441.4+j23.86	4.284e-08
3	2442.3+j23.82	2.929e-08
4	2505.5+j23.56	3.806e-07
5	2505.0+j23.56	7.312e-07

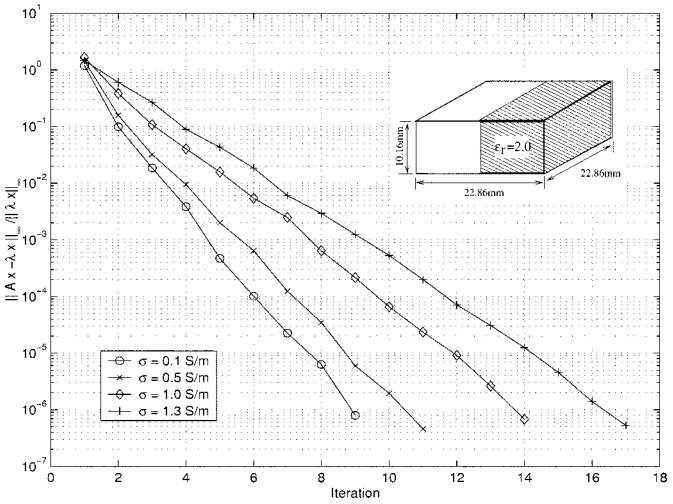

Fig. 3. Convergence of the dominant mode of a half-filled resonator for various values of the conductivity of the filling dielectric.

Since the structure is lossless, the block Lanczos algorithm may also be used. In order to demonstrate the impact on loss of orthogonality in the Lanczos process on the calculated eigenvectors, the algorithm was applied for the calculation of the first few lowest eigenfrequencies of the cylindrical cavity.

The seven lowest eigenfrequencies obtained after nine iterations are listed in Table II. The occurrence of two spurious modes is attributed to the loss of orthogonality in the Lanczos process. Also, compared with the Arnoldi algorithm, the Lanczos process exhibited slower convergence.

For the case where the dielectric rod inside the cylindrical cavity is lossy, the FEM eigenvalues are obtained using the block Arnoldi technique. For the specific application, the conductivity of the rod was taken to be $\sigma = 0.1$ S/m. Once again, the five lowest eigenmodes are of interest; thus, the

TABLE IV
EIGENFREQUENCY OF THE DOMINANT MODE OF A PARTIALLY FILLED RECTANGULAR RESONATOR

σ(S/m)	Analytical	Calculated	Residual
0.1	7.379+j0.354	7.378+j0.353	7.934e-07
0.5	7.236+j1.819	7.236+j1.813	4.594e-07
1.0	6.579+j3.864	6.585+j3.852	6.756e-07
1.3	5.711+j5.197	5.724+j5.182	5.280e-07

expansion frequency was taken to be $f_0 = 1.2$ GHz. Convergence for all five modes was achieved after 14 iterations. The total CPU time was 73 s. The calculated eigenfrequencies are shown in Table III. Their convergence is depicted in the bottom plot of Fig. 2. In order to examine the dependence of the convergence behavior of the algorithm as the expansion frequency point changes, the same problem was solved by setting $f_0 = 1.0$ GHz. This time, the number of required iterations for convergence of the five lowest eigenfrequencies was 16. However, the same eigenfrequencies were obtained.

As a final example, the half-filled rectangular resonator shown in the insert of Fig. 3 was analyzed. The base is square of a side of 22.86 mm, while the height is 10.16 mm. The average grid size of the FEM mesh is 2.49 mm, and the number of unknowns is 2734. The calculated dominant eigenfrequency (in gigahertz) for different values of the conductivity of the dielectric is compared to its analytical value in Table IV. Also, Fig. 3 depicts the impact of loss on convergence.

VII. CONCLUSIONS

In conclusion, in this paper, efficient and reliable algorithms have been proposed and validated for an FEM-based solution of electromagnetic eigenvalue problems involving both lossless and lossy media. For lossy media in particular, the proposed FEM model is based on the discretization of the system of Maxwell's curl equations. The major attribute of this formulation, referred to as the $\vec{E} - \vec{B}$ formulation, is that the resulting discrete eigenvalue problem is linear. The apparent increased computational complexity due to the increase in the number of degrees of freedom (since both $\vec{E}$ and $\vec{B}$ are discretized) is avoided by taking advantage of a special relationship between the spaces used for the expansions of the two fields and by means of a modification of the Lanczos or Arnoldi algorithm used for the extraction of a subset of the dominant eigenvalues. From the two algorithms, the Arnoldi process was found to be the most robust and algorithmically simple.

While the FEM formulations used are free of spurious ac modes, the occurrence of spurious dc modes (i.e., modes that exhibit nonzero divergence) is possible and must be avoided since it slows convergence and impacts accuracy. The proposed methodology for their avoidance is based on the selection of divergence-free initial vectors and the explicit elimination from the generated Lanczos–Arnoldi vectors of any spurious dc parts caused by roundoff error through the introduction of appropriate static fields in the lossless portions of the computational domain. The validity, computational efficiency, accuracy, and robustness of the proposed algorithms were demonstrated through a series of numerical experiments. Finally, it is mentioned that the algorithms proposed in this paper can also be used in conjunction with high-order finite-element approximations.

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